Mathematics 24 Midterm 1 Take-Home Part
Spring 2013
Due Monday, April 22 at 6:00 pm in Room 207 Kemeny

This examination consists of three problems. You are to do your own work and not discuss the exam with anyone. For sources you may use the textbook, your homework or your class notes. You may cite a result without proof if it appears in either (1) the assigned reading in the text (2) the assigned homework (3) your class notes. If you cannot completely solve a problem, you should indicate how far you have gotten.

Important: Write on one side of the paper and show your work. Messy and barely legible papers will not be considered. Give reasons, but try to keep your solutions short.

1. (20 points) Let V be a vector space, let $S \subseteq V$ be a subset and let $\langle S \rangle$ be the subspace spanned by S. Prove

$$\langle S \rangle = \bigcap_{S \subseteq W} W,$$

where the intersection is of all subspaces W which contain S. (You may assume without proof that the right-hand side is a vector space.)

2. (15 points) Let $T : V \to V$ be a linear transformation and define $T^2 : V \to V$ by $T^2(v) = T(T(v))$ for all $v \in V$. Assume that $T^2 = T$ and prove

$$V = N(T) \oplus W,$$

where W is the subspace defined by

$$W = \{v \in V \mid T(v) = v\}.$$

See the second definition on p. 22 for the two conditions for a direct sum. Hint: consider $v - T(v)$.

3. (20 points) Let V be a vector space and $T : V \to V$ a linear transformation. Prove

1. If $V = R(T) + N(T)$, then $V = R(T) \oplus N(T)$.
2. If $R(T) \cap N(T) = \{0\}$, then $V = R(T) \oplus N(T)$.